

An L^1 -estimate for certain spectral multipliers associated with the Ornstein–Uhlenbeck operator

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ABSTRACT. We study a class of spectral multipliers $\phi(L)$ for the Ornstein–Uhlenbeck operator L arising from the Gaussian measure on \mathbb{R}^n and find a sufficient condition for integrability of $\phi(L)f$ in terms of the admissible conical square function and a maximal function.

1. Introduction

On the Euclidean space \mathbb{R}^n , the *Ornstein–Uhlenbeck operator*

$$L = -\frac{1}{2}\Delta + x \cdot \nabla$$

is associated with the Gaussian measure

$$d\gamma(x) = \pi^{-n/2} e^{-|x|^2} dx$$

by the Dirichlet form

$$\int_{\mathbb{R}^n} (Lf)g d\gamma = \frac{1}{2} \int_{\mathbb{R}^n} \nabla f \cdot \nabla g d\gamma, \quad f, g \in C_0^\infty(\mathbb{R}^n).$$

It has discrete spectrum $\sigma(L) = \{0, 1, 2, \dots\}$ on $L^2(\gamma)$, and an orthonormal basis of eigenfunctions is given by *Hermite polynomials* h_β , $\beta \in \mathbb{N}^n$, so that $Lh_\beta = |\beta|h_\beta$. Moreover, it generates a (positive) diffusion semigroup on $L^2(\gamma)$ which can be expressed as

$$e^{-tL}f(x) = \int_{\mathbb{R}^n} M_t(x, y)f(y) d\gamma(y), \quad t > 0,$$

by means of the *Mehler kernel* (see [12])

$$M_t(x, y) = \frac{1}{(1 - e^{-2t})^{n/2}} \exp\left(-\frac{e^{-t}}{1 - e^{-2t}}|x - y|^2\right) \exp\left(\frac{e^{-t}}{1 + e^{-t}}(|x|^2 + |y|^2)\right).$$

The semigroup is contractive on $L^p(\gamma)$ for each $1 \leq p \leq \infty$, and acts conservatively so that $e^{-tL}1 = 1$. Therefore, E. M. Stein’s theory [13] is applicable in studying the boundedness of spectral multipliers $\phi(L)$ defined as $\phi(L)h_\beta = \phi(|\beta|)h_\beta$ for $\beta \in \mathbb{N}^n$. More precisely, [13, Corollary 3, p. 121] guarantees $L^p(\gamma)$ -boundedness with $1 < p < \infty$, for any spectral multiplier of ‘Laplace transform type’, i.e. of the form

$$\phi(\lambda) = \lambda \int_0^\infty \Phi(t)e^{-t\lambda} dt, \quad \lambda \geq 0,$$

where $\Phi : (0, \infty) \rightarrow \mathbb{C}$ is bounded. In particular, the imaginary powers $L^{i\tau}$, $\tau \in \mathbb{R}$, arising from $\Phi(t) = t^{-i\tau}/\Gamma(1 - i\tau)$ are bounded on $L^p(\gamma)$. The general theory was improved by

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M. Cowling [3] who showed that, for a given $p \in (1, \infty)$, $\phi(L)$ is bounded on $L^p(\gamma)$ as soon as ϕ extends analytically to a sector of angle greater than $\pi|1/p - 1/2|$. (See also the more recent development [2].)

The L^1 -theory in the Gaussian setting is quite problematic. Although finite linear combinations of Hermite polynomials are dense in $L^1(\gamma)$, the spectral projections onto their eigenspaces are not $L^1(\gamma)$ -bounded. Moreover, tLe^{-tL} is bounded (uniformly) on $L^p(\gamma)$ only when $1 < p < \infty$ (see [7, Chapter 5]).

A Gaussian weak $(1, 1)$ -type estimate for spectral multipliers of Laplace transform type was established by J. García-Cuerva et al. [5]. Moreover, in [4] they showed that requiring analyticity of ϕ in a sector of angle smaller than $\arcsin|1 - 2/p|$ will not alone suffice for boundedness of $\phi(L)$ on $L^p(\gamma)$. Observing that $\arcsin|1 - 2/p| \rightarrow \pi/2$ as $p \rightarrow 1$ is in line with the fact that the spectrum of L on $L^1(\gamma)$ is the (closed) right half-plane. Furthermore, $L^1(\gamma)$ -boundedness of dilation invariant spectral multipliers for L was characterised in [6, Theorem 3.5 (ii)].

The main obstruction in developing a metric theory of Hardy spaces in the Gaussian setting arises from the fact that the rapidly decaying measure γ is non-doubling, that is, for every $t > 0$

$$\frac{\gamma(B(x, 2t))}{\gamma(B(x, t))} \longrightarrow \infty, \quad \text{as } |x| \rightarrow \infty.$$

G. Mauceri and S. Meda overcame this problem in [10] and developed an atomic theory for a Gaussian Hardy space which relies of the fact that the Gaussian measure behaves well locally with respect to the *admissibility function*

$$m(x) = \min(1, |x|^{-1}), \quad x \in \mathbb{R}^n.$$

Indeed, γ is doubling on families of ‘admissible’ Euclidean balls

$$\mathcal{B}_\alpha = \{B(x, t) : 0 < t \leq \alpha m(x)\}, \quad \alpha \geq 1,$$

in the sense that for all $\lambda \geq 2$ we have

$$(1) \quad \gamma(\lambda B) \leq e^{4\lambda^2 \alpha^2} \gamma(B), \quad B \in \mathcal{B}_\alpha.$$

Other natural objects that are suitable for defining Hardy spaces, namely maximal functions and square functions, were studied in the Gaussian setting by J. Maas, J. van Neerven, and P. Portal. In [8] they considered (a version¹ of) the *admissible conical square function*

$$Sf(x) = \left(\int_0^{2m(x)} \frac{1}{\gamma(B(x, t))} \int_{B(x, t)} |t^2 Le^{-t^2 L} f(y)|^2 d\gamma(y) \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

and showed that it is controlled by a non-tangential semigroup maximal function. The converse inequality was presented in [11] along with a proof that the Riesz transform satisfies $\|\nabla L^{-1/2} f\|_1 \lesssim \|Sf\|_1 + \|f\|_1$. The benefit of conical objects (as opposed to vertical ones) is the applicability of tent space theory, which in the Gaussian setting was initiated in [9] and further developed by A. Amenta and the author in [1].

The aim of this paper is to examine the decomposition method presented in [11] and to see what kind of L^1 -estimates one can obtain for spectral multipliers $\phi(L)f$ in terms of the admissible conical square function Sf and other relevant objects. The hope is that these considerations will help in developing a fully satisfactory theory of Gaussian Hardy spaces.

THEOREM. *Let*

$$\phi(\lambda) = \int_0^\infty \Phi(t)(t\lambda)^2 e^{-t\lambda} \frac{dt}{t}, \quad \lambda \geq 0,$$

¹They have $t\nabla e^{-t^2 L}$ instead of $t^2 Le^{-t^2 L}$.

where $\Phi : (0, \infty) \rightarrow \mathbb{C}$ is twice continuously differentiable and satisfies

$$\sup_{0 < t < \infty} (|\Phi(t)| + t|\Phi'(t)| + t^2|\Phi''(t)|) + \int_1^\infty (|\Phi'(t)| + t|\Phi''(t)|) dt < \infty.$$

Then, for all $f \in L^1(\gamma)$, we have

$$\|\phi(L)f\|_1 \lesssim \|Sf\|_1 + \|f\|_1 + \|(1 + \log_+ |\cdot|) Mf\|_1,$$

where

$$Mf(x) = \sup_{\varepsilon m(x)^2 < t \leq 1} |e^{-tL} f(x)|, \quad x \in \mathbb{R}^n,$$

and $\varepsilon > 0$ does not depend on f .

REMARKS. Several remarks are in order:

- (1) The term $\|(1 + \log_+ |\cdot|) Mf\|_1$ is highly undesirable for two reasons. Firstly, the maximal operator M is of a non-admissible kind in the sense that it is not restricted to times $t \lesssim m(\cdot)$. Secondly, the weight factor $(1 + \log_+ |\cdot|)$, which arises from the admissibility function m , seems problematic. However, it is difficult to see how the appearance of the term could be avoided. Notice, nevertheless, that $\|(1 + \log_+ |\cdot|) Mf\|_1$ is finite at least if $f \in L^p(\gamma)$ with $1 < p < \infty$.
- (2) The operators in the theorem above are special kind of Laplace type multipliers;

$$\phi(\lambda) = \int_0^\infty \Phi(t)(t\lambda)^2 e^{-t\lambda} \frac{dt}{t} = \lambda \int_0^\infty (\Phi(t) + t\Phi'(t)) e^{-t\lambda} dt, \quad \lambda \geq 0,$$

and therefore bounded on $L^p(\gamma)$ when $1 < p < \infty$. Note that if, in addition, we had

$$\int_0^1 (|\Phi'(t)| + t|\Phi''(t)|) dt < \infty,$$

then $\phi(L)$ would be bounded even on $L^1(\gamma)$. Indeed, using integration by parts we have

$$\phi(L)f = -\Phi(0)f + \int_0^\infty (2\Phi'(t) + t\Phi''(t)) e^{-tL} f dt$$

so that $\|e^{-tL} f\|_1 \leq \|f\|_1$ implies

$$\|\phi(L)f\|_1 \lesssim \left(|\Phi(0)| + \int_0^\infty (|\Phi'(t)| + t|\Phi''(t)|) dt \right) \|f\|_1.$$

- (3) An example of a multiplier satisfying the conditions of the theorem is the localized imaginary power arising from $\Phi(t) = t^{i\tau} \chi(t)$, where $\tau \in \mathbb{R}$ and χ is a smooth cutoff with, say, $1_{(0,1]} \leq \chi \leq 1_{(0,2]}$. Observe that for $0 < t \leq 1$ we have $|\Phi'(t)| \approx t^{-1}$ and $|\Phi''(t)| \approx t^{-2}$ so that

$$\int_0^1 (|\Phi'(t)| + t|\Phi''(t)|) dt = \infty.$$

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2. Proof of the theorem

Strategy. The proof of the theorem follows the decomposition method from [11]. Let us begin by introducing a discretized version of the admissibility function

$$\tilde{m}(x) = \begin{cases} 1, & |x| < 1, \\ 2^{-k}, & 2^{k-1} \leq |x| < 2^k, \quad k \geq 1, \end{cases}$$

and write $\tilde{\mathcal{B}}_\alpha$ for the associated family of admissible balls. From $\tilde{m} \leq m \leq 2\tilde{m}$ it follows that $\tilde{\mathcal{B}}_\alpha \subset \mathcal{B}_\alpha \subset \tilde{\mathcal{B}}_{2\alpha}$. This discretization is relevant for Proposition 7.

We define the *Gaussian tent space* adapted to this new admissibility function as the space $\mathfrak{t}^1(\gamma)$ of functions u on the *admissible region* $D = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : 0 < t < \tilde{m}(y)\}$ for which

$$\|u\|_{\mathfrak{t}^1(\gamma)} = \int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |u(y, t)|^2 \frac{d\gamma(y) dt}{t\gamma(B(y, t))} \right)^{1/2} d\gamma(x) < \infty.$$

Here $\Gamma(x) = \{(y, t) \in D : |y - x| < t\}$ is an admissible cone at $x \in \mathbb{R}^n$.

The main theorem of [1] guarantees that every $u \in \mathfrak{t}^1(\gamma)$ admits a decomposition into ‘atoms’ a_k so that

$$u = \sum_k \lambda_k a_k, \quad \text{with} \quad \sum_k |\lambda_k| \approx \|u\|_{\mathfrak{t}^1(\gamma)}.$$

Recall that *atom* is a function a on D associated with a ball $B \in \tilde{\mathcal{B}}_5$ for which $\text{supp } a \subset B \times (0, r_B)$ and

$$\left(\int_0^{r_B} \|a(\cdot, t)\|_2^2 \frac{dt}{t} \right)^{1/2} \leq \gamma(B)^{-1/2}.$$

For such a function, $\|a\|_{\mathfrak{t}^1(\gamma)} \lesssim 1$.

Let then ϕ and Φ be as in Theorem and let f be a polynomial. For any $\delta, \delta' > 0$ and $\kappa \geq 1$ we can decompose $\phi(L)f$ into three parts as follows:

$$\begin{aligned} \phi(L)f &= c_{\delta, \delta'} \int_0^\infty \Phi((\delta' + \delta)t^2) (t^2 L)^2 e^{-(\delta' + \delta)t^2 L} f \frac{dt}{t} \\ &= c_{\delta, \delta'} \left(\int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} u(\cdot, t) \frac{dt}{t} \right. \\ &\quad + \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} (1_{D^c}(\cdot, t) t^2 L e^{-\delta t^2 L} f) \frac{dt}{t} \\ &\quad \left. + \int_{\tilde{m}(\cdot)/\kappa}^\infty \tilde{\Phi}(t^2) (t^2 L)^2 e^{-(\delta' + \delta)t^2 L} f \frac{dt}{t} \right) \\ &=: c_{\delta, \delta'} (\pi_1 u + \pi_2 f + \pi_3 f), \end{aligned}$$

where $u(\cdot, t) = 1_D(\cdot, t) t^2 L e^{-\delta t^2 L} f$ and $\tilde{\Phi}(t) = \Phi((\delta' + \delta)t)$.

Now

$$\|\phi(L)f\|_1 \leq |c_{\delta, \delta'}| (\|\pi_1 u\|_1 + \|\pi_2 f\|_1 + \|\pi_3 f\|_1),$$

and the proof consists of estimating these three terms separately for sufficiently small $\delta > \delta' > 0$ and large enough $\kappa \geq 1$.

Analysis of the three parts. Proposition 2 deals with

$$\pi_1 u = \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} u(\cdot, t) \frac{dt}{t}$$

and relies on the following L^2 - L^2 -off diagonal estimate (cf. [11, Proposition 4.2] and [14]).

LEMMA 1. *There exists a constant $c_{od} > 0$ such that for $j = 0, 1$ we have*

$$\|1_{E'}(tL)^j e^{-tL} 1_E\|_{2 \rightarrow 2} \lesssim \exp\left(-\frac{d(E, E')^2}{c_{od}t}\right), \quad t > 0,$$

whenever $E, E' \subset \mathbb{R}^n$.

PROPOSITION 2. *Let $\kappa \geq 1$ and $0 < \delta \leq 1$. For sufficiently small $\delta' > 0$ we have $\|\pi_1 u\|_1 \lesssim \|u\|_{\mathfrak{t}^1(\gamma)}$. Moreover, the function $u(\cdot, t) = 1_D(\cdot, t)t^2 L e^{-\delta t^2 L} f$ satisfies $\|u\|_{\mathfrak{t}^1(\gamma)} \lesssim \|Sf\|_1$.*

PROOF. By the atomic decomposition, it suffices to show that $\|\pi_1 a\|_1 \lesssim 1$ for any atom a associated with a ball $B \in \tilde{\mathcal{B}}_5$. Let us consider the annuli $C_k(B) = 2^{k+1}B \setminus 2^k B$ for $k \geq 1$, and $C_0(B) = 2B$. By Hölder's inequality we have

$$\begin{aligned} \|\pi_1 a\|_1 &= \left\| \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} a(\cdot, t) \frac{dt}{t} \right\|_1 \\ (2) \quad &\leq \sum_{k=0}^{\infty} \gamma(2^{k+1}B)^{1/2} \left\| 1_{C_k(B)} \int_0^{r_B \wedge 2^{-k-1}/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} a(\cdot, t) \frac{dt}{t} \right\|_2. \end{aligned}$$

We estimate the norms on the right hand side of (2) by pairing with a $g \in L^2(\gamma)$ and relying on the assumption that Φ is bounded:

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \int_0^{r_B \wedge 2^{-k-1}/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} a(\cdot, t) \frac{dt}{t} g d\gamma \right| \\ &= \left| \int_0^{r_B \wedge 2^{-k-1}/\kappa} \int_B a(\cdot, t) \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} g d\gamma \frac{dt}{t} \right| \\ &\lesssim \left(\int_0^{r_B} \|a(\cdot, t)\|_2^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^{r_B} \|1_B t^2 L e^{-\delta' t^2 L} g\|_2^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Now, for g supported in $C_0(B) = 2B$ we have

$$\begin{aligned} \left(\int_0^{r_B} \|1_B t^2 L e^{-\delta' t^2 L} g\|_2^2 \frac{dt}{t} \right)^{1/2} &= \left(\sum_{\beta \in \mathbb{N}^n} |\langle g, h_\beta \rangle|^2 \int_0^{r_B} (t^2 |\beta|)^2 e^{-2\delta' t^2 |\beta|} \frac{dt}{t} \right)^{1/2} \\ &\lesssim \|g\|_2. \end{aligned}$$

When $k \geq 1$ we have $d(C_k(B), B) \geq (2^k - 1)r_B \geq 2^{k-1}r_B$ and so, by Lemma 1, it follows that for $0 < t \leq r_B$,

$$\|1_B t^2 L e^{-\delta' t^2 L} 1_{C_k(B)}\|_{2 \rightarrow 2} \lesssim \exp\left(-\frac{4^{k-1} r_B^2}{c_{od} \delta' t^2}\right) \lesssim \exp\left(-\frac{4^{k-2}}{c_{od} \delta'}\right) \left(\frac{t}{r_B}\right)^{1/2}.$$

Hence, for g supported in $C_k(B)$, $k \geq 1$, we have

$$\left(\int_0^{r_B} \|1_B t^2 L e^{-\delta' t^2 L} g\|_2^2 \frac{dt}{t} \right)^{1/2} \lesssim \exp\left(-\frac{4^{k-2}}{c_{od} \delta'}\right) \|g\|_2.$$

We have therefore shown that, for $k \geq 0$,

$$\left\| 1_{C_k(B)} \int_0^{r_B} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} a(\cdot, t) \frac{dt}{t} \right\|_2 \lesssim \exp\left(-\frac{4^{k-2}}{c_{od} \delta'}\right) \gamma(B)^{-1/2}.$$

According to the doubling inequality (1), we have $\gamma(2^{k+1}B)^{1/2} \lesssim e^{2 \cdot 4^{k+1} \cdot 25} \gamma(B)^{1/2}$ and therefore

$$\|\pi_1 a\|_1 \lesssim \sum_{k=0}^{\infty} \gamma(2^{k+1}B)^{1/2} \exp\left(-\frac{4^{k-2}}{c_{od} \delta'}\right) \gamma(B)^{-1/2} \lesssim \sum_{k=0}^{\infty} \exp\left(50 \cdot 4^{k+1} - \frac{4^{k-2}}{c_{od} \delta'}\right) \lesssim 1$$

as soon as $\delta' < 1/(3200c_{od})$. This proves the first claim.

For the second claim, let $u(\cdot, t) = 1_D(\cdot, t)t^2Le^{-\delta t^2L}f$. We perform a change of variable, $\delta t^2 = s^2$, i.e. $t = s/\sqrt{\delta}$ so that

$$\begin{aligned} (y, t) \in \Gamma(x) &\Leftrightarrow |y - x| < t < \tilde{m}(y) \\ &\Leftrightarrow |y - x| < s/\sqrt{\delta} < \tilde{m}(y) \\ &\Leftrightarrow (y, s) \in \Gamma'_{1/\sqrt{\delta}}(x) := \{(y, s) \in D' : |y - x| < s/\sqrt{\delta}\}, \end{aligned}$$

where $D' := \{(y, s) \in \mathbb{R}^n \times (0, \infty) : s < \sqrt{\delta}\tilde{m}(y)\}$. Now, change of aperture in the Gaussian tent space on D' (see [1, Corollary 3.5]) guarantees that

$$\begin{aligned} \|u\|_{\mathfrak{t}^1(\gamma)} &= \int_{\mathbb{R}^n} \left(\iint_{\Gamma(x)} |t^2Le^{-\delta t^2L}f(y)|^2 \frac{d\gamma(y)dt}{t\gamma(B(y, t))} \right)^{1/2} d\gamma(x) \\ &= \int_{\mathbb{R}^n} \left(\iint_{\Gamma'_{1/\sqrt{\delta}}(x)} |\delta^{-1}s^2Le^{-s^2L}f(y)|^2 \frac{d\gamma(y)ds}{s\gamma(B(y, s/\sqrt{\delta}))} \right)^{1/2} d\gamma(x) \\ &\lesssim \int_{\mathbb{R}^n} \left(\iint_{\Gamma'(x)} |s^2Le^{-s^2L}f(y)|^2 \frac{d\gamma(y)ds}{s\gamma(B(y, s))} \right)^{1/2} d\gamma(x). \end{aligned}$$

We then observe (see [9, Lemma 2.3]) that for any $x, y \in \mathbb{R}^n$, $|y - x| < s < m(y)$ implies $s < 2m(x)$, and therefore

$$\Gamma'(x) \subset \Gamma(x) \subset \bigcup_{0 < s < 2m(x)} B(x, s) \times \{s\}.$$

Moreover, $\gamma(B(y, s)) \approx \gamma(B(x, s))$ when $|y - x| < s < \delta\tilde{m}(y)$, and hence

$$\iint_{\Gamma'(x)} |s^2Le^{-s^2L}f(y)|^2 \frac{d\gamma(y)ds}{s\gamma(B(y, s))} \lesssim \int_0^{2m(x)} \frac{1}{\gamma(B(x, s))} \int_{B(x, s)} |s^2Le^{-s^2L}f(y)|^2 d\gamma(y) \frac{ds}{s}$$

for every $x \in \mathbb{R}^n$, which shows that $\|u\|_{\mathfrak{t}^1(\gamma)} \lesssim \|Sf\|_1$ as required. \square

For π_2 and π_3 (more precisely, for Proposition 5 and Lemma 6) we need the following two lemmas concerning pointwise kernel estimates.

LEMMA 3. *Let $j = 0, 1$. For all $x, y \in \mathbb{R}^n$ we have the pointwise kernel estimate*

$$|t^j \partial_t^j M_t(x, y)| \lesssim t^{-n/2} \exp\left(-\frac{|x - y|^2}{8t}\right) \exp\left(\frac{|x|^2 + |y|^2}{2}\right), \quad 0 < t \leq 1.$$

As a consequence, for all $0 < t \leq 1$ we have

$$\|1_{E'}(tL)^j e^{-tL} 1_E\|_{1 \rightarrow \infty} \lesssim t^{-n/2} \exp\left(-\frac{d(E, E')^2}{8t}\right) \sup_{\substack{x \in E \\ y \in E'}} \exp\left(\frac{|x|^2 + |y|^2}{2}\right),$$

whenever $E, E' \subset \mathbb{R}^n$.

PROOF. For $0 < t \leq 1$ we have the elementary estimates

$$\frac{1}{1 - e^{-2t}} \approx \frac{1}{t}, \quad \frac{1}{4t} \leq \frac{e^{-t}}{1 - e^{-2t}} \leq \frac{1}{2t}, \quad \frac{1}{8} \leq \frac{e^{-t}}{1 + e^{-t}} \leq \frac{1}{2}$$

and the case $j = 0$ follows immediately:

$$\begin{aligned} M_t(x, y) &= \frac{1}{(1 - e^{-2t})^{n/2}} \exp\left(-\frac{e^{-t}}{1 - e^{-2t}}|x - y|^2\right) \exp\left(\frac{e^{-t}}{1 + e^{-t}}(|x|^2 + |y|^2)\right) \\ &\lesssim t^{-n/2} \exp\left(-\frac{|x - y|^2}{4t}\right) \exp\left(\frac{|x|^2 + |y|^2}{2}\right). \end{aligned}$$

For $j = 1$ we calculate:

$$\partial_t M_t(x, y) = \left(-\frac{ne^{-2t}}{1 - e^{-2t}} + |x - y|^2 \frac{e^{-t}(1 + e^{-2t})}{(1 - e^{-2t})^2} - (|x|^2 + |y|^2) \frac{e^{-t}}{(1 + e^{-t})^2} \right) M_t(x, y).$$

Using the previous case $j = 0$ we then see that

$$\begin{aligned} |t\partial_t M_t(x, y)| &\lesssim \left(1 + \frac{|x - y|^2}{t} + |x|^2 + |y|^2\right) M_t(x, y) \\ &\lesssim t^{-n/2} \exp\left(-\frac{|x - y|^2}{8t}\right) \exp\left(\frac{|x|^2 + |y|^2}{2}\right). \end{aligned}$$

The consequence is also immediate: for any $x \in E'$ we have

$$\begin{aligned} |(tL)^j e^{-tL} f(x)| &\lesssim t^{-n/2} \int_E \exp\left(-\frac{|x - y|^2}{8t}\right) \exp\left(\frac{|x|^2 + |y|^2}{2}\right) |f(y)| d\gamma(y) \\ &\lesssim t^{-n/2} \exp\left(-\frac{d(E, E')^2}{8t}\right) \sup_{y \in E} \exp\left(\frac{|x|^2 + |y|^2}{2}\right) \int_E |f(y)| d\gamma(y). \end{aligned}$$

□

LEMMA 4. *For α large enough there exists a constant $c > 0$ such that for all $x, y \in \mathbb{R}^n$ and all $0 < t \leq 1$ we have*

$$M_{t/\alpha}(x, y) \lesssim \exp\left(-\frac{|x - y|^2}{ct}\right) \exp\left(\alpha t \min(|x|^2, |y|^2)\right) M_t(x, y),$$

and, consequently,

$$|t\partial_t M_{t/\alpha}(x, y)| \lesssim \exp\left(\alpha t \min(|x|^2, |y|^2)\right) M_t(x, y).$$

PROOF. An alternative way to express the Mehler kernel is (see [12])

$$M_t(x, y) = \frac{1}{(1 - e^{-2t})^{n/2}} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) e^{|y|^2}.$$

By [11, Lemma 3.4] for α large enough we have for all $x, y \in \mathbb{R}^n$ and all $0 < t \leq 1$ that

$$\exp\left(-\frac{|e^{-t/\alpha}x - y|^2}{1 - e^{-2t/\alpha}}\right) \leq \exp\left(-2\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) \exp\left(\frac{t^2 \min(|x|^2, |y|^2)}{1 - e^{-2t/\alpha}}\right).$$

Therefore

$$M_{t/\alpha}(x, y) \lesssim \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) \exp\left(\frac{t^2 \min(|x|^2, |y|^2)}{1 - e^{-2t/\alpha}}\right) M_t(x, y),$$

where, by symmetry, the first exponential factor can be replaced by

$$\exp\left(-\frac{\max(|e^{-t}x - y|^2, |x - e^{-t}y|^2)}{1 - e^{-2t}}\right).$$

The first claim now follows because for all $x, y \in \mathbb{R}^n$ and all $0 < t \leq 1$ we have

$$|x - y|^2 \lesssim \max(|e^{-t}x - y|^2, |x - e^{-t}y|^2).$$

In order to see this, let us assume, with no loss of generality, that $|x| \leq |y|$, and show that $|x - y|^2 \lesssim |e^{-t}x - y|^2$. Then

$$\begin{aligned} |x - y|^2 &\leq e(e^{-t}|x|^2 - 2e^{-t}x \cdot y + e^{-t}|y|^2) \\ &= e(e^{-t}|x|^2 - (1 - e^{-t})|y|^2 - 2e^{-t}x \cdot y + |y|^2), \end{aligned}$$

where

$$e^{-t}|x|^2 - (1 - e^{-t})|y|^2 \leq e^{-2t}|x|^2,$$

because $|x| \leq |y|$. Indeed,

$$e^{-t}|x|^2 - (1 - e^{-t})|x|^2 - e^{-2t}|x|^2 = (2e^{-t} - 1 - e^{-2t})|x|^2,$$

where $2e^{-t} - 1 - e^{-2t} \leq 0$ for all $t > 0$.

The second claim now follows from the first one since

$$\begin{aligned} |t\partial_t M_{t/\alpha}(x, y)| &\lesssim \left(1 + \frac{|x - y|^2}{t} + |x|^2 + |y|^2\right) M_{t/\alpha}(x, y) \\ &\lesssim \exp\left(\alpha t \min(|x|^2, |y|^2)\right) M_t(x, y). \end{aligned}$$

Here the first inequality is obtained as in the proof of Lemma 3 (case $j = 1$). \square

Let us then consider

$$\pi_2 f = \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} (1_{D^c}(\cdot, t) t^2 L e^{-\delta t^2 L} f) \frac{dt}{t}.$$

PROPOSITION 5. *Let $\kappa \geq 4$. For sufficiently small $\delta > \delta' > 0$ we have $\|\pi_2 f\|_1 \lesssim \|f\|_1$.*

PROOF. We begin by observing that if $t \leq \tilde{m}(x)/4$ and $t > 2^{-k-1}$ for some $k \geq 2$, then $|x| < 2^{k-2}$. Moreover, if $t \geq \tilde{m}(y)$ and $t \leq 2^{-k}$, then $|y| \geq 2^{k-1}$.

We then decompose $\pi_2 f$ (using boundedness of Φ) as follows:

$$\begin{aligned} \|\pi_2 f\|_1 &= \left\| \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} (1_{D^c}(\cdot, t) t^2 L e^{-\delta t^2 L} f) \frac{dt}{t} \right\|_1 \\ (3) \quad &\lesssim \sum_{k=2}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \|1_{B(0, 2^{k-2})} t^2 L e^{-\delta' t^2 L} (1_{\mathbb{R}^n \setminus B(0, 2^{k-1})} t^2 L e^{-\delta t^2 L} f)\|_1 \frac{dt}{t} \\ &\leq \sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \|1_{B(0, 2^{k-2})} t^2 L e^{-\delta' t^2 L} (1_{C_{k+l-1}} t^2 L e^{-\delta t^2 L} f)\|_1 \frac{dt}{t}, \end{aligned}$$

where $C_{k+l-1} := B(0, 2^{k+l-1}) \setminus B(0, 2^{k+l-2})$.

First, by Lemma 4, we choose a $\delta > 0$ such that for all $0 < t \leq 1$ we have

$$|t^2 L e^{-\delta t^2 L} f(x)| \lesssim \exp\left(\frac{t^2 |x|^2}{\delta}\right) |e^{-tL} f(x)|, \quad x \in \mathbb{R}^n.$$

Hence, for $2^{-k-1} < t \leq 2^{-k}$ we have

$$\|1_{C_{k+l-1}} t^2 L e^{-\delta t^2 L} f\|_1 \lesssim \exp\left(\frac{4^{-k} \cdot 4^{k+l-1}}{\delta}\right) \|e^{-tL} f\|_1 \lesssim \exp\left(\frac{4^{l-1}}{\delta}\right) \|f\|_1.$$

Then, since the distance between $B(0, 2^{k-2})$ and C_{k+l-1} is at least 2^{k+l-3} , we have, by Lemma 3, for $2^{-k-1} < t \leq 2^{-k}$ that

$$\begin{aligned} \|1_{B(0, 2^{k-2})} t^2 L e^{-\delta' t^2 L} 1_{C_{k+l-1}}\|_{1 \rightarrow 1} &\lesssim t^{-n} \exp\left(-\frac{4^{k+l-3}}{8\delta' t^2}\right) \exp\left(\frac{4^{k-2} + 4^{k+l-1}}{2}\right) \\ &\lesssim 2^{kn} \exp\left(-\frac{4^{2k+l-5}}{\delta'} + 4^{k+l-1}\right). \end{aligned}$$

Combining the two estimates we see that for $2^{-k-1} < t \leq 2^{-k}$ we have

$$\begin{aligned} &\|1_{B(0, 2^{k-2})} t^2 L e^{-\delta' t^2 L} (1_{C_{k+l-1}} t^2 L e^{-\delta t^2 L} f)\|_1 \\ &\lesssim 2^{kn} \exp\left(-\frac{4^{2k+l-5}}{\delta'} + 4^{k+l-1} + \frac{4^{l-1}}{\delta}\right) \|f\|_1 \\ &= 2^{kn} \exp\left(-4^{k+l+1} \left(\frac{4^{k-6}}{\delta'} - 4^{-2} - \frac{4^{-2}}{\delta}\right)\right) \|f\|_1 \\ &\lesssim \exp(-4^{k+l}) \|f\|_1, \end{aligned}$$

where in the last step we chose $\delta' < \delta$ small enough.

The right-hand side of (3) is therefore dominated by

$$\sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \exp(-4^{k+l}) \|f\|_1 \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{t} \lesssim \|f\|_1.$$

□

LEMMA 6. *For any $\alpha > 0$ we have*

$$\|(e^{-tL}f)|_{t=\tilde{m}(\cdot)^2/\alpha}\|_1 \lesssim \|f\|_1.$$

Moreover, for α large enough we have

$$\|(tLe^{-tL}f)|_{t=\tilde{m}(\cdot)^2/\alpha}\|_1 \lesssim \|f\|_1.$$

PROOF. Write $C_0 = B(0, 1)$ and $C_k = B(0, 2^k) \setminus B(0, 2^{k-1})$ for $k \geq 1$. Moreover, let $C_0^* = B(0, 2)$, $C_1^* = B(0, 4)$, and $C_k^* = B(0, 2^{k+1}) \setminus B(0, 2^k)$ for $k \geq 2$.

We first show that for any $\alpha > 0$,

$$(4) \quad \|(e^{-tL}f)|_{t=\tilde{m}(\cdot)^2/\alpha}\|_1 \lesssim \|f\|_1.$$

Denote $\varepsilon = 1/\alpha$ for notational convenience. For $x \in C_k$ we have $\tilde{m}(x)^2 = 4^{-k}$ and hence

$$\|(e^{-tL}f)|_{t=\varepsilon\tilde{m}(\cdot)^2}\|_1 = \sum_{k=0}^{\infty} \|1_{C_k} e^{-\varepsilon 4^{-k} L} f\|_1.$$

We split f into $1_{C_k^*} f$ and $1_{\mathbb{R}^n \setminus C_k^*} f$, and first estimate

$$\sum_{k=0}^{\infty} \|1_{C_k} e^{-\varepsilon 4^{-k} L} (1_{C_k^*} f)\|_1 \leq \sum_{k=0}^{\infty} \|1_{C_k^*} f\|_1 \lesssim \|f\|_1.$$

Fixing an integer N for which $8\varepsilon \leq 4^N$, we use the trivial estimate for $k = 0, 1, \dots, N+3$:

$$\|1_{C_k} e^{-\varepsilon 4^{-k} L} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_1 \leq \|f\|_1.$$

For $k \geq N+4$ we have the decomposition

$$\mathbb{R}^n \setminus C_k^* = B(0, 2^{k-2}) \cup \bigcup_{l=2}^{\infty} C_{k+l}.$$

Observing that $d(C_k, B(0, 2^{k-2})) = 2^{k-2}$ we obtain, by Lemma 3,

$$\begin{aligned} \|1_{C_k} e^{-\varepsilon 4^{-k} L} 1_{B(0, 2^{k-2})}\|_{1 \rightarrow 1} &\lesssim 2^{kn} \exp\left(-\frac{4^{k-2}}{8\varepsilon 4^{-k}}\right) \exp\left(\frac{4^k + 4^{k-2}}{2}\right) \\ &\leq 2^{kn} \exp(-4^{2k-2-N} + 4^k) \\ &\lesssim \exp(-4^k). \end{aligned}$$

Furthermore, since $d(C_k, C_{k+l}) = 2^{k+l-2}$, Lemma 3 implies that

$$\begin{aligned} \|1_{C_k} e^{-\varepsilon 4^{-k} L} 1_{C_{k+l}}\|_{1 \rightarrow 1} &\lesssim 2^{kn} \exp\left(-\frac{4^{k+l-2}}{8\varepsilon 4^{-k}}\right) \exp\left(\frac{4^k + 4^{k+l}}{2}\right) \\ &\leq 2^{kn} \exp(-4^{2k+l-2-N} + 4^{k+l}) \\ &\lesssim \exp(-4^{k+l}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=N+4}^{\infty} \|1_{C_k} e^{-\varepsilon 4^{-k} L} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_1 &= \sum_{k=N+4}^{\infty} \left(\|1_{C_k} e^{-\varepsilon 4^{-k} L} (1_{B(0, 2^{k-2})} f)\|_1 \right. \\ &\quad \left. + \sum_{l=2}^{\infty} \|1_{C_k} e^{-\varepsilon 4^{-k} L} (1_{C_{k+l}} f)\|_1 \right) \\ &\lesssim \sum_{k=N+4}^{\infty} \left(\exp(-4^k) \|f\|_1 + \sum_{l=2}^{\infty} \exp(-4^{k+l}) \|f\|_1 \right) \\ &\lesssim \|f\|_1. \end{aligned}$$

We have now proven (4) which includes the first case from the statement of the lemma.

The second case follows by using Lemma 4, which guarantees that there exists an $\alpha > 0$ such that for all $x, y \in \mathbb{R}^n$

$$\left| (t\partial_t M_t(x, y))|_{t=\tilde{m}(x)^2/\alpha} \right| \lesssim \exp\left(\alpha \frac{\tilde{m}(x)^2}{\alpha} |x|^2\right) M_{\tilde{m}(x)^2}(x, y) \lesssim M_{\tilde{m}(x)^2}(x, y).$$

Then

$$\|(tLe^{-tL}f)|_{t=\tilde{m}(\cdot)^2/\alpha}\|_1 \lesssim \|(e^{-tL}f)|_{t=\tilde{m}(\cdot)^2}\|_1 \lesssim \|f\|_1.$$

□

Finally, we turn to

$$\pi_3 f = \int_{\tilde{m}(\cdot)/\kappa}^{\infty} \tilde{\Phi}(t^2)(t^2 L)^2 e^{-(\delta'+\delta)t^2 L} f \frac{dt}{t}.$$

PROPOSITION 7. *Let $0 < \delta, \delta' \leq 1/2$. For κ large enough we have $\|\pi_3 f\|_1 \lesssim \|f\|_1 + \|(1 + \log_+ |\cdot|) Mf\|_1$, where $Mf(x) = \sup_{\varepsilon m(x)^2 < t \leq 1} |e^{-tL} f(x)|$ and $\varepsilon > 0$ does not depend on f .*

PROOF. Integrating by parts we obtain

$$\begin{aligned} & \int_{\tilde{m}(\cdot)/\kappa}^{\infty} \tilde{\Phi}(t^2)(t^2 L)^2 e^{-(\delta'+\delta)t^2 L} f \frac{dt}{t} \\ &= c \int_{\tilde{m}(\cdot)^2/\kappa^2}^{\infty} \tilde{\Phi}(t) t \partial_t^2 e^{-(\delta'+\delta)tL} f dt \\ &= c \left[\tilde{\Phi}(t) t \partial_t e^{-(\delta'+\delta)tL} f \right]_{t=\tilde{m}(\cdot)^2/\kappa^2}^{\infty} + c' \int_{\tilde{m}(\cdot)^2/\kappa^2}^{\infty} (\tilde{\Phi}(t) + t\tilde{\Phi}'(t)) \partial_t e^{-(\delta'+\delta)tL} f dt. \end{aligned}$$

Repeating for the last term we get

$$\begin{aligned} & \int_{\tilde{m}(\cdot)^2/\kappa^2}^{\infty} (\tilde{\Phi}(t) + t\tilde{\Phi}'(t)) \partial_t e^{-(\delta'+\delta)tL} f dt \\ &= c \left[(\tilde{\Phi}(t) + t\tilde{\Phi}'(t)) e^{-(\delta'+\delta)tL} f \right]_{t=\tilde{m}(\cdot)^2/\kappa^2}^{\infty} + c' \int_{\tilde{m}(\cdot)^2/\kappa^2}^{\infty} (2\tilde{\Phi}'(t) + t\tilde{\Phi}''(t)) e^{-(\delta'+\delta)tL} f dt. \end{aligned}$$

Now, having assumed that $\sup_{0 < t < \infty} (|\Phi(t)| + t|\Phi'(t)|) < \infty$, we may use Lemma 6 to choose κ large enough so that

$$\left\| \left[\tilde{\Phi}(t) t \partial_t e^{-(\delta'+\delta)tL} f \right]_{t=\tilde{m}(\cdot)^2/\kappa^2}^{\infty} \right\|_1 \lesssim \|(tLe^{-(\delta'+\delta)tL} f)|_{t=\tilde{m}(\cdot)^2/\kappa^2}\|_1 \lesssim \|f\|_1$$

and

$$\left\| \left[(\tilde{\Phi}(t) + t\tilde{\Phi}'(t)) e^{-(\delta'+\delta)tL} f \right]_{t=\tilde{m}(\cdot)^2/\kappa^2}^{\infty} \right\|_1 \lesssim \|(e^{-(\delta'+\delta)tL} f)|_{t=\tilde{m}(\cdot)^2/\kappa^2}\|_1 \lesssim \|f\|_1.$$

Moreover,

$$\left\| \int_1^{\infty} (2\tilde{\Phi}'(t) + t\tilde{\Phi}''(t)) e^{-(\delta'+\delta)tL} f dt \right\|_1 \lesssim \int_1^{\infty} (|\tilde{\Phi}'(t)| + t|\tilde{\Phi}''(t)|) \|e^{-(\delta'+\delta)tL} f\|_1 dt \lesssim \|f\|_1.$$

Finally, having assumed that $\sup_{0 < t < \infty} (t|\Phi'(t)| + t^2|\Phi''(t)|) < \infty$, we get

$$\begin{aligned} \left| \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 (2\tilde{\Phi}'(t) + t\tilde{\Phi}''(t)) e^{-(\delta'+\delta)tL} f dt \right| &\lesssim \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 (|\tilde{\Phi}'(t)| + t|\tilde{\Phi}''(t)|) |e^{-(\delta'+\delta)tL} f| dt \\ &\lesssim \sup_{\varepsilon m(\cdot)^2 < t \leq 1} |e^{-tL} f| \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 \frac{dt}{t} \\ &\lesssim (1 + \log_+ |\cdot|) Mf, \end{aligned}$$

where $\varepsilon > 0$ is chosen small enough depending on δ, δ' and κ .

□

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